

STOCHASTIC DIFFERENTIAL EQUATION INVOLVING WIENER PROCESS AND FRACTIONAL BROWNIAN MOTION WITH HURST INDEX $H > 1/2$

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ABSTRACT. We consider a mixed stochastic differential equation driven by possibly dependent fractional Brownian motion and Brownian motion. Under mild regularity assumptions on the coefficients, it is proved that the equation has a unique solution.

INTRODUCTION

Fractional Brownian motion (fBm) with a Hurst parameter $H \in (0, 1)$ is defined formally as a continuous centered Gaussian process $B_t^H = \{B_t^H, t \geq 0\}$ with the covariance $EB_t^H B_s^H = 1/2(s^{2H} + t^{2H} - |t - s|^{2H})$. For $H > 1/2$ it exhibits a property of long-range dependence, which makes it a popular model for long-range dependence in natural sciences, financial mathematics etc. For this reason, equations driven by fractional Brownian motion have been an object of intensive study during the last decade.

There are two principal ways to define an integral with respect to fractional Brownian motion.

One possibility is Skorokhod, or divergence integral introduced in the fractional Brownian setting in [3]. However this definition is not very practical: it is based on Wick rather than usual products, and unlike Brownian case, in the fractional Brownian case this makes difference when integrating non-anticipating functions because of dependence of increments. This makes this definition worthless for most applications (most notably, those in financial mathematics). Moreover, it is impossible to solve stochastic differential equations with such integral except the cases of additive or multiplicative noise; the latter case was considered in [9].

Another approach is a pathwise integral, defined first in [16] for fBm with $H > 1/2$ as a Young integral. The papers [7, 13, 14] were the first to prove existence and uniqueness of stochastic differential equations involving such integrals. Later the pathwise approach was extended with the help of Lyons' rough path theory to the case of arbitrary H in [1] where also unique solvability of equations with $H > 1/4$ was proved. Numerical methods for pathwise stochastic differential equations with fBm were considered in [11, 12, 2, 4].

In this paper we focus on the following mixed stochastic differential equation involving Wiener process and fractional Brownian motion with Hurst index $H \in$

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$(1/2, 1)$:

$$(1) \quad X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s + \int_0^t c(s, X_s) dB_s^H, \quad t \in [0, T],$$

where the integral w.r.t. Wiener process is the standard Itô integral, and the integral w.r.t. fBm is pathwise generalized Lebesgue–Stieltjes, or Young integral. The motivation to consider such equations comes e.e. from financial applications, where Brownian motion as a model is inappropriate because of the lack of memory, and fractional Brownian motion with $H > 1/2$ is too smooth.

Unique solvability of (1) was proved in [8] for time-independent coefficients and zero drift, [10] for $H \in (3/4, 1)$ and bounded coefficients, and in [5] for any $H > 1/2$, but under the assumption that W and B^H are independent. We generalize the latter result proving that (1) has a unique solution for any $H \in (1/2, 1)$ with W and B^H possibly dependent. The paper is organized as follows. In Section 1, we give necessary definitions and main hypotheses. In Section 2, we define Euler approximations of (1) and establish useful facts for them. In Section 3, we prove fundamental property of Euler approximations, and Section 4 contains the main result about existence and uniqueness of solution to (1).

1. BASIC DEFINITIONS AND ASSUMPTIONS

1.1. Fractional derivatives, integrals and norms. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space equipped with a filtration satisfying standard assumptions. Denote $\{W_t, t \in [0, T]\}$ a standard \mathcal{F}_t -Wiener process, and $\{B_t^H, t \geq 0\}$ an fBm adapted to the filtration \mathcal{F}_t .

To integrate with respect to fractional Brownian motion, we use the generalized (fractional) Lebesgue–Stieltjes integral (see [13, 16]). It is defined as follows.

Consider two continuous functions f and g , defined on some interval $[a, b] \subset \mathbb{R}$. For $\alpha \in (0, 1)$ define fractional derivatives

$$\begin{aligned} (D_{a+}^\alpha f)(x) &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(u)}{(x-u)^{1+\alpha}} du \right) 1_{(a,b)}(x), \\ (D_{b-}^{1-\alpha} g)(x) &= \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x) - g(u)}{(x-u)^{2-\alpha}} du \right) 1_{(a,b)}(x). \end{aligned}$$

Assume that $D_{a+}^\alpha f \in L_p[a, b]$, $D_{b-}^{1-\alpha} g \in L_q[a, b]$ for some $p \in (1, 1/\alpha)$, $q = p/(p-1)$.

Under these assumptions, the generalized (fractional) Lebesgue–Stieltjes, or Young integral $\int_a^b f(x) dg(x)$ is defined as

$$\int_a^b f(x) dg(x) = e^{i\pi\alpha} \int_a^b (D_{a+}^\alpha f)(x) (D_{b-}^{1-\alpha} g)(x) dx.$$

It was shown in [15] that for any $\alpha \in (1-H, 1)$ there exists the fractional derivative $D_{b-}^{1-\alpha} B^H \in L_\infty[a, b]$. Hence, for f with $D_{a+}^\alpha f \in L_1[a, b]$ we can define the integral w.r.t. fBm according to this formula:

$$(2) \quad \int_a^b f_s dB_s^H = e^{i\pi\alpha} \int_a^b (D_{a+}^\alpha f)(x) (D_{b-}^{1-\alpha} B^H)(x) dx.$$

In view of this, we will consider the following two norms for $\alpha \in (1 - H, 1/2)$:

$$\begin{aligned} \|f\|_{2,\alpha,[a,b]}^2 &= \int_a^b \left(|f(s)| + \int_a^s |f(s) - f(z)| (s-z)^{-1-\alpha} dz \right)^2 (s^{-\alpha} + (t-s)^{-\alpha-1/2}) ds, \\ \|f\|_{\infty,\alpha,[a,b]} &= \sup_{s \in [a,b]} \left(|f(s)| + \int_a^s |f(s) - f(z)| (s-z)^{-1-\alpha} dz \right). \end{aligned}$$

It is clear that $\|f\|_{2,\alpha,[a,b]} \leq C_{\alpha,a,b} \|f\|_{\infty,\alpha,[a,b]}$. Throughout the paper there will be no ambiguity about α , so for the sake of shortness we will denote $\|f\|_{x,t} = \|f\|_{x,\alpha,[0,t]}$, where $x \in \{2, \infty\}$.

1.2. Estimates for stochastic integrals and increments. Recall that the classical Garsia–Rodemich–Rumsey inequality [?] states that for a function $f \in C([0, T])$ and any $p > 0$, $\theta > 1/p$

$$(3) \quad \sup_{0 \leq v < u \leq T} \frac{|f(u) - f(v)|}{(u-v)^{\theta-1/p}} \leq C_{\alpha,p,\theta} \left(\int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\theta p+1}} dx dy \right)^{1/p}.$$

Setting in this inequality for $\eta \in (0, 1/2)$ $p = 2/\eta$, $\theta = (1 - \eta)/2$, we get that for any $t > 0$, $u, v \in [0, t]$

$$(4) \quad |W_u - W_v| \leq K_t^{W,\eta} |u - v|^{1/2-\eta},$$

where

$$(5) \quad K_t^{W,\eta} = C_\eta \left(\int_0^t \int_0^t \frac{|W_x - W_y|^{2/\eta}}{|x-y|^{1/\eta}} dx dy \right)^{\eta/2},$$

C_η is a nonrandom constant.

Similarly, we get for any $\eta \in (0, H)$

$$(6) \quad |B_u^H - B_v^H| \leq K_t^{B,\eta} |u - v|^{H-\eta},$$

where

$$K_t^{B,\eta} = C_{H,\eta} \left(\int_0^t \int_0^t \frac{|B_x^H - B_y^H|^{2/\eta}}{|x-y|^{2H/\eta}} dx dy \right)^{\eta/2}.$$

Hence it is easy to deduce that for any $\alpha \in (1 - H, 1/2)$, $\varepsilon < \alpha + H - 1$

$$\sup_{0 \leq u < v \leq t} |(D_{v-}^{1-\alpha} B^H)(u)| \leq C_{\alpha,H,\varepsilon} K_t^{B,\varepsilon}.$$

Thus, thanks to (2), the stochastic integral with respect to fBm admits the following estimate:

$$(7) \quad \left| \int_u^v f(s) dB_s^H \right| \leq C_\alpha K_t^{B,\varepsilon} \int_u^v \left(|f(s)| (s-u)^{-\alpha} + \int_u^s |f(s) - f(z)| (s-z)^{-\alpha-1} dz \right) ds$$

for any $\alpha \in (1 - H, 1/2)$, $t > 0$, $u \leq v \leq t$ and any f such that the right-hand side of this inequality is finite.

1.3. Assumptions. In what follows we will assume the following standard hypotheses.

(A) *Linear growth:* for any $t \in [0, T]$ and any $x \in \mathbb{R}$

$$|a(t, x)| + |c(t, x)| \leq K(1 + |x|).$$

(B) *Lipschitz continuity of a, b :* for any $t \in [0, T]$ and $x, y \in \mathbb{R}$

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y|.$$

(C) *Hölder continuity in time:* the function $c(t, x)$ is differentiable in x and there exists $\beta \in (1 - H, 1)$ such that for any $s, t \in [0, T]$ and any $x \in \mathbb{R}$

$$|a(s, x) - a(t, x)| + |b(s, x) - b(t, x)| + |c(s, x) - c(t, x)| + |\partial_x c(s, x) - \partial_x c(t, x)| \leq K|s - t|^\beta.$$

(D) *Lipschitz continuity of $\partial_x c$:* for any $t \in [0, T]$ and any $x, y \in \mathbb{R}$

$$|\partial_x c(t, x) - \partial_x c(t, y)| \leq K|x - y|.$$

(E) *Boundedness of b and $\partial_x c$:* for any $T \in [0, T]$ and $x \in \mathbb{R}$

$$|b(t, x)| + |\partial_x c(t, x)| \leq K.$$

Here K is a constants independent of x, y, s and t .

2. AUXILIARY PROPERTIES OF EULER APPROXIMATIONS

For $n \geq 1$ consider the following partition of the fixed interval $[0, T] : \{0 = \nu_0 < \nu_1 < \dots < \nu_n = T, \delta = T/n\}$, $\nu_k = k\delta$.

Consider Euler approximation for equation (1):

$$X_{\nu_{k+1}}^\delta = X_{\nu_k}^\delta + a(\nu_k, X_{\nu_k}^\delta)(\nu_{k+1} - \nu_k) + b(\nu_k, X_{\nu_k}^\delta)(W_{\nu_{k+1}} - W_{\nu_k}) + c(\nu_k, X_{\nu_k}^\delta)(B_{\nu_{k+1}}^H - B_{\nu_k}^H),$$

with $X_{\nu_0}^\delta = X_0$.

Set $t_u^\delta = \max\{\nu_n : \nu_n \leq u\}$ and define continuous interpolation by

$$X_u^\delta = X_{t_u^\delta}^\delta + a(t_u^\delta, X_{t_u^\delta}^\delta)(u - t_u^\delta) + b(t_u^\delta, X_{t_u^\delta}^\delta)(W_u - W_{t_u^\delta}) + c(t_u^\delta, X_{t_u^\delta}^\delta)(B_u^H - B_{t_u^\delta}^H),$$

or, in the integral form,

$$(8) \quad X_u^\delta = X_0 + \int_0^u a(t_s^\delta, X_{t_s^\delta}^\delta)ds + \int_0^u b(t_s^\delta, X_{t_s^\delta}^\delta)dW_s + \int_0^u c(t_s^\delta, X_{t_s^\delta}^\delta)dB_s^H.$$

Observe that the estimates (4) and (6) together with our main hypotheses imply that

$$(9) \quad |X_s^\delta - X_{t_s^\delta}^\delta| \leq CK_s^\eta (s - t_s^\delta)^{1/2-\eta} (1 + |X_{t_s^\delta}^\delta|).$$

with $K_s^\eta = K_s^{B,\eta} + K_s^{W,\eta}$. For $N \geq 1$ define a stopping time $\tau_N = \inf \{t : K_t^\eta \geq N\} \wedge T$ and a stopped process $X_t^{\delta,N} = X_{t \wedge \tau_N}^\delta$.

The following lemma provides an estimate of Euler approximations, which is essential to establishing the main result, because it is independent of δ .

Lemma 2.1. For $\alpha \in (1 - H, \frac{1}{2} \wedge \beta)$, $p > 0$, $N \geq 1$

$$\sup_\delta E \left[\|X^{\delta,N}\|_{\infty,T}^p \right] < \infty.$$

Proof. Take any $\eta \in (0, 1/2 - \alpha)$. In this proof by C we will denote a generic constant, which may depend on $p, \alpha, \eta, T, \lambda, X_0$ and the constants from the main hypotheses, but is independent of N and δ .

Denote $\|f\|_t = |f(t)| + \int_0^t |f(t) - f(s)| (t-s)^{-1-\alpha} ds$, so that $\|f\|_{\infty, t} = \sup_{s \in [0, t]} \|f\|_s$.

Now fix any $t < \tau_N$. Write

$$\begin{aligned} \|X^\delta\|_t &\leq |X_0| + I_a(t) + I_b(t) + I_c(t) := \\ &|X_0| + \left\| \int_0^\cdot a(t_s^\delta, X_{t_s}^\delta) ds \right\|_t + \left\| \int_0^\cdot b(t_s^\delta, X_{t_s}^\delta) dW_s \right\|_t + \left\| \int_0^\cdot c(t_s^\delta, X_{t_s}^\delta) dB_s^H \right\|_t. \end{aligned}$$

Estimate

$$\begin{aligned} I_a(t) &\leq \int_0^t |a(t_s^\delta, X_{t_s}^\delta)| ds + \int_0^t \int_s^t |a(t_v^\delta, X_{t_v}^\delta)| dv (t-s)^{-1-\alpha} ds \\ &\leq C \left(\int_0^t (1 + |X_{t_s}^\delta|) ds + \int_0^t \int_s^t (1 + |X_{t_v}^\delta|) dv (t-s)^{-1-\alpha} ds \right) \\ &\leq C \left(1 + \int_0^t \|X^\delta\|_{\infty, v} ds + \int_0^t \|X^\delta\|_{\infty, v} (t-v)^{-\alpha} dv \right) \\ &\leq C \left(1 + \int_0^t \|X^\delta\|_{\infty, v} (t-v)^{-\alpha} dv \right). \end{aligned}$$

Further, write

$$I_c(t) \leq I'_c + I''_c := \left| \int_0^t c(t_s^\delta, X_{t_s}^\delta) dB_s^H \right| + \int_0^t \left| \int_s^t c(t_v^\delta, X_{t_v}^\delta) dB_v^H \right| (t-s)^{-1-\alpha} ds.$$

By (7) and (9) (recall that $t < \tau_n$)

$$\begin{aligned} I'_c &\leq CK_t^\eta \int_0^t \left(|c(t_s^\delta, X_{t_s}^\delta)| s^{-\alpha} + \int_0^s |c(t_s^\delta, X_{t_s}^\delta) - c(t_z^\delta, X_{t_z}^\delta)| (s-z)^{-1-\alpha} dz \right) ds \\ &\leq CN \int_0^t \left((1 + \|X^\delta\|_{\infty, s}) s^{-\alpha} + \int_0^{t_s^\delta} ((t_s^\delta - t_z^\delta)^\beta + |X_{t_s}^\delta - X_{t_z}^\delta|) (s-z)^{-1-\alpha} dz \right) ds \\ &\leq CN \left(1 + \int_0^t \|X^\delta\|_{\infty, s} s^{-\alpha} ds + \int_0^t \int_0^{t_s^\delta} ((s-z)^\beta + (z-t_z^\delta)^\beta + \delta^\beta) (s-z)^{-1-\alpha} dz ds \right. \\ &\quad \left. + \int_0^t \int_0^{t_s^\delta} (|X_s^\delta - X_z^\delta| + |X_s^\delta - X_{t_s}^\delta| + |X_z^\delta - X_{t_z}^\delta|) (s-z)^{-1-\alpha} dz \right) \\ &\leq CN \left(1 + \int_0^t \|X^\delta\|_{\infty, s} s^{-\alpha} ds + t^{\beta-\alpha+1} + \delta^{\beta-\alpha} + \int_0^t \|X^\delta\|_{\infty, s} ds \right. \\ &\quad \left. + CN \int_0^t \int_0^{t_s^\delta} \left[(1 + |X_{t_s}^\delta|) (s-t_s^\delta)^{1/2-\eta} + (1 + |X_{t_z}^\delta|) (z-t_z^\delta)^{1/2-\eta} \right] (s-z)^{-1-\alpha} dz ds \right) \\ &\leq CN^2 \left(1 + \int_0^t \|X^\delta\|_{\infty, s} s^{-\alpha} ds + \int_0^t \|X^\delta\|_{\infty, s} (s-t_s^\delta)^{1/2-\eta-\alpha} ds + J_c \right). \end{aligned}$$

Here the inequality $\int_0^t \int_0^{t_s^\delta} (z-t_z^\delta)^\delta (s-z)^{-1-\alpha} dz ds < C\delta^{\beta-\alpha}$ is obtained similarly to the following estimate of J_c , in which we change the order of integration noting

that $z < t_s^\delta \iff t_z^\delta + \delta \leq s$:

$$\begin{aligned}
J_c &= \int_0^t \int_0^{t_s^\delta} (1 + |X_{t_z^\delta}^\delta|) (z - t_z^\delta)^{1/2-\eta} (s - z)^{-1-\alpha} dz ds \\
&= \int_0^{t_t^\delta} (1 + |X_{t_z^\delta}^\delta|) (z - t_z^\delta)^{1/2-\eta} \int_{t_z^\delta + \delta}^t (s - z)^{-1-\alpha} ds dz \\
&= \int_0^{t_t^\delta} (1 + |X_{t_z^\delta}^\delta|) (z - t_z^\delta)^{1/2-\eta} (t_z^\delta + \delta - z)^{-\alpha} ds dz \\
&= \sum_{k=1}^{[t/\delta]} (1 + |X_{\nu_{k-1}}^\delta|) \int_{\nu_{k-1}}^{\nu_k} (z - \nu_{k-1})^{1/2-\eta} (\nu_k - z)^{-\alpha} dz \\
&= \sum_{k=1}^{[t/\delta]} (1 + |X_{\nu_{k-1}}^\delta|) \delta \cdot \delta^{1/2-\eta-\alpha} B(3/2-\eta, 1-\alpha) \leq C \left(1 + \int_0^t \|X^\delta\|_{\infty, s} ds \right).
\end{aligned}$$

Hence we can write

$$I'_c \leq CN^2 \left(1 + \int_0^t \|X^\delta\|_{\infty, s} s^{-\alpha} ds \right).$$

Further, denoting $c_s = c(t_s, X_{t_s^\delta}^\delta)$,

$$\begin{aligned}
I''_c &\leq CN \int_0^t \int_s^t \left(|c_v| (v - s)^{-\alpha} + \int_s^v |c_v - c_z| (v - z)^{-1-\alpha} dz \right) dv (t - s)^{-1-\alpha} ds \\
&\leq CN \left(\int_0^t \int_0^v (1 + \|X^\delta\|_{\infty, v}) (v - s)^{-\alpha} (t - s)^{-1-\alpha} ds dv \right. \\
&\quad \left. + \int_0^t \int_0^v \left((t_v^\delta - t_z^\delta)^\beta + |X_{t_v^\delta}^\delta - X_{t_z^\delta}^\delta| \right) (v - z)^{-1-\alpha} (t - z)^{-\alpha} dz dv \right) \\
&\leq CN(H_1 + H_2 + H_3),
\end{aligned}$$

where

$$\begin{aligned}
H_1 &\leq \int_0^t (1 + \|X^\delta\|_{\infty,v}) \int_0^v (v-s)^{-\alpha} (t-s)^{-1-\alpha} ds dv \\
&\leq C \int_0^t (1 + \|X^\delta\|_{\infty,v}) (t-v)^{-2\alpha} dv \leq C + C \int_0^t \|X^\delta\|_{\infty,v} (t-v)^{-2\alpha} dv, \\
H_2 &\leq \int_0^t \int_0^{t_v^\delta} ((v-z)^\beta + (z-t_z^\delta)^\beta) (v-z)^{-1-\alpha} (t-z)^{-\alpha} dz dv \\
&\leq C + \int_0^t \int_0^{t_v^\delta} (z-t_z^\delta)^\beta (v-z)^{-1-\alpha} (t-z)^{-\alpha} dz dv, \\
H_3 &\leq \int_0^t \int_0^{t_v^\delta} (|X_v^\delta - X_z^\delta| + |X_v^\delta - X_{t_v^\delta}^\delta| + |X_z^\delta - X_{t_z^\delta}^\delta|) (v-z)^{-1-\alpha} (t-z)^{-\alpha} dz dv \\
&\leq C \int_0^t \int_0^v |X_v^\delta - X_z^\delta| (v-z)^{-1-\alpha} dz (t-v)^{-\alpha} dv \\
&\quad + CN \int_0^t (v-t_v^\delta)^{1/2-\eta-\alpha} (1 + \|X^\delta\|_{\infty,v}) (t-v)^{-\alpha} dv \\
&\quad + CN \int_0^t \int_0^{t_v^\delta} (z-t_z^\delta)^{1/2-\eta} (1 + |X_{t_z^\delta}^\delta|) (v-z)^{-\alpha-1} (t-z)^{-\alpha} dz dv \\
&\leq CN \int_0^t (1 + \|X^\delta\|_{\infty,v}) (t-v)^{-\alpha} dv + CN H'_3.
\end{aligned}$$

Here H'_3 is the integral appearing in the penultimate line; we skip the estimation of the last integral in H_2 , because it is analogous to the estimation of H'_3 , but somewhat simpler.

Write (abbreviating $r = 1/2 - \eta$ and changing the order of integration)

$$\begin{aligned}
H'_3 &= \int_0^{t_t^\delta} (z-t_z^\delta)^r (1 + |X_{t_z^\delta}^\delta|) (t-z)^{-\alpha} \int_{t_z^\delta+\delta}^t (v-z)^{-\alpha-1} dv dz \\
&\leq C \int_0^{t_t^\delta} (1 + |X_{t_z^\delta}^\delta|) (z-t_z^\delta)^r (t-z)^{-\alpha} (t_z^\delta+\delta-z)^{-\alpha} dz \\
&= C \sum_{k=1}^{[t/\delta]} (1 + |X_{\nu_{k-1}}^\delta|) \int_{\nu_{k-1}}^{\nu_k} (z-\nu_{k-1})^r (\nu_k-z)^{-\alpha} (t-z)^{-\alpha} dz \\
(10) \quad &\leq \sum_{k=1}^{[t/\delta]-1} (1 + |X_{\nu_{k-1}}^\delta|) (t-\nu_k)^{-\alpha} \int_{\nu_{k-1}}^{\nu_k} (z-\nu_{k-1})^r (\nu_k-z)^{-\alpha} dz \\
&\quad + (1 + |X_{t_t^\delta-\delta}^\delta|) \delta^r \int_{t_t^\delta-\delta}^{t_t^\delta} (t_t^\delta-z)^{-\alpha} (t-z)^{-\alpha} dz \\
&\leq \sum_{k=1}^{[t/\delta]-1} (1 + \|X^\delta\|_{\infty,\nu_k}) (t-\nu_k)^{-\alpha} \delta s + (1 + \|X^\delta\|_{\infty,t_t^\delta-\delta}) \delta^r (t-t_t^\delta)^{1-2\alpha} \\
&\leq \int_0^t (1 + \|X^\delta\|_{\infty,s}) (t-s)^{-2\alpha} ds.
\end{aligned}$$

As a result,

$$I_c(t) \leq CN^2 \left(1 + \int_0^t \|X^\delta\|_{\infty,s} (s^{-\alpha} + (t-s)^{-2\alpha}) ds \right).$$

Adding the estimate for $I_a(t)$, we get for $t \leq \tau_N$

$$\begin{aligned} \|X^\delta\|_t &\leq CN^2 \left(1 + \int_0^t \|X^\delta\|_{\infty,s} (s^{-\alpha} + (t-s)^{-2\alpha}) ds \right) + I_b(t) \\ &\leq CN^2 \left(1 + \sup_{s \in [0,T]} I_b(s) + t^{-2\alpha} \int_0^t \|X^\delta\|_{\infty,s} s^{-2\alpha} (t-s)^{-2\alpha} ds \right). \end{aligned}$$

Therefore, by the generalized Gronwall lemma [13, Lemma 7.6], for $t \leq \tau_N$

$$\|X^\delta\|_{\infty,t} \leq CN^2 \sup_{s \in [0,T]} I_b(s) \exp \left\{ C t N^{2/(1-2\alpha)} \right\} \leq C_N \sup_{s \in [0,T]} I_b(s).$$

Putting $t = T \wedge \tau_N$ and using the obvious fact that $\|X^{\delta,N}\|_{\infty,T} = \|X^\delta\|_{\infty,T \wedge \tau_N}$, we get

$$\|X^{\delta,N}\|_{\infty,T} \leq C_N \sup_{s \in [0,T]} I_b(s).$$

So it remains to prove that $E[\sup_{s \in [0,T]} I_b(s)^p]$ is bounded uniformly in δ . Write

$$E[\sup_{s \in [0,T]} I_b(s)^p] \leq I'_b + I''_b,$$

where, denoting $b_s^\delta = b(t_s^\delta, X(t_s^\delta))$,

$$\begin{aligned} I'_b &= E \left[\sup_{t \in [0,T]} \left| \int_0^t b_s^\delta dW_s \right|^p \right] \leq C \int_0^T E \left[|b(t_s^\delta, X(t_s^\delta))|^p \right] ds \leq C, \\ I''_b &= E \left[\sup_{t \in [0,T]} \left(\int_0^t \left| \int_s^t b(t_z^\delta, X(t_z^\delta)) dW_z \right| (t-s)^{-1-\alpha} ds \right)^p \right]. \end{aligned}$$

It follows from the Garsia–Rodemich–Rumsey inequality that $\left| \int_s^t b(t_z^\delta, X(t_z^\delta)) dW_z \right| \leq \xi_\delta |t-s|^{1/2-\eta}$, where

$$\xi_\delta = C \left(\int_0^T \int_0^T \frac{|\int_x^y b_v^\delta dW_v|^{2/\eta}}{|x-y|^{1/\eta}} dx dy \right)^{\eta/2}$$

We have

$$E[\xi_\delta^p] \leq C \int_0^T \int_0^T \frac{E \left[|\int_x^y b_v^\delta dW_v|^{2p/\eta} \right]}{|x-y|^{p/\eta}} dx dy C \int_0^T \int_0^T \frac{|\int_x^y E[(b_v^\delta)^2] dv|^{p/\eta}}{|x-y|^{p/\eta}} dx dy \leq C,$$

whence (recalling that $1/2 - \eta > \alpha$)

$$I''_b = E[\xi_\delta^p] \sup_{t \in [0,T]} \left(\int_0^t (t-s)^{-1/2-\eta-\alpha} ds \right)^p \leq C,$$

as required. \square

3. A FUNDAMENTAL PROPERTY OF THE SEQUENCE OF EULER APPROXIMATIONS

Consider a pair of partitions defined as $\{\nu_i = iT/n, 0 \leq i \leq n\}$ and $\{\theta_j = jT/(n2^m), 0 \leq j \leq n2^m\}$.

Denote the diameters of these partitions, correspondingly, by $\delta = T/n$ and $\mu = \delta 2^{-m}$.

Let $t_u^\delta = \max\{\nu_n : \nu_n \leq u\}$ and $t_u^\mu = \max\{\theta_k : \theta_k \leq u\}$. Continuous interpolations of corresponding Euler approximations can be written in the integral form:

$$(11) \quad \begin{aligned} X_u^\delta &= X_0^\delta + \int_0^u a(t_s^\delta, X_{t_s^\delta}^\delta) ds + \int_0^u b(t_s^\delta, X_{t_s^\delta}^\delta) dW_s + \int_0^u c(t_s^\delta, X_{t_s^\delta}^\delta) dB_s^H, \\ X_u^\mu &= X_0^\mu + \int_0^u a(t_s^\mu, X_{t_s^\mu}^\mu) ds + \int_0^u b(t_s^\mu, X_{t_s^\mu}^\mu) dW_s + \int_0^u c(t_s^\mu, X_{t_s^\mu}^\mu) dB_s^H. \end{aligned}$$

Define, as before, a stopping time $\tau_N = \inf\{t : K_t^\eta \geq N\} \wedge T$ and $X_t^{\delta,N} = X_{t \wedge \tau_N}^\delta$, $X_t^{\mu,N} = X_{t \wedge \tau_N}^\mu$. For $R \geq 1$ define $B_t^{R,\delta,\mu} = \left\{ \|X^\delta\|_{\infty,t} + \|X^\mu\|_{\infty,t} \leq R \right\}$.

Theorem 3.1. *Let $\alpha \in (1 - H, \kappa)$, where $\kappa = \frac{1}{2} \wedge \beta$. Then for any $0 < \eta < \kappa - \alpha$ and $N, R \geq 1$ the following estimate holds*

$$E \left[\|X^{\delta,N} - X^{\mu,N}\|_{2,T}^2 \mathbb{I}_{B_T^{R,\delta,\mu}} \right] \leq M_{R,N} \delta^{2(\kappa - \alpha - \varepsilon)},$$

where the constant $M_{R,N}$ is independent of δ, μ .

Moreover, a similar estimate is valid for $E \left[\sup_{t \in [0,T]} |X_t^{\delta,N} - X_t^{\mu,N}|^2 \mathbb{I}_{B_T^{R,\delta,\mu}} \right]$.

Proof. As in the proof of Lemma 2.1, by C we will denote a generic constant, which may depend on α, η, T, X_0 , the constants from the main hypotheses, but is independent of δ and μ, N and R . For the sake of shortness for a process Z define $Z_{t,s} = Z_t - Z_s$ and denote $r = 1/2 - \eta$, $h(t,s) = (t-s)^{-1-\alpha}$, $g(t,s) = s^{-\alpha} + (t-s)^{-\alpha-1/2}$, $\mathbb{I}_t = \mathbb{I}_{B_t^{R,\delta,\mu}}$.

It follows from (11) that

$$(12) \quad \begin{aligned} X_u^{\delta,N} - X_u^{\mu,N} &= \int_0^{u(N)} a_\Delta(s) ds + \int_0^{u(N)} b_\Delta(s) dW_s + \int_0^{u(N)} c_\Delta(s) dB_s^H \\ &=: \mathcal{I}_a(u) + \mathcal{I}_b(u) + \mathcal{I}_c(u), \end{aligned}$$

where $d_\Delta(s) := d(t_s^\delta, X_{t_s^\delta}^{\delta,N}) - d(t_s^\mu, X_{t_s^\mu}^{\mu,N})$, $d \in \{a, b, c\}$. Due to our hypotheses, on

$$(13) \quad \begin{aligned} |d_\Delta(s)| &\leq C \left(|t_s^\delta - t_s^\mu|^\beta + |X_{t_s^\delta}^{\delta,N} - X_{t_s^\mu}^{\mu,N}| \right) \leq C \left((s - t_s^\delta)^\beta + |X_{t_s^\delta}^{\delta,N}| + |X_{t_s^\mu}^{\delta,N} - X_{t_s^\mu}^{\mu,N}| \right) \\ &\leq C \left((s - t_s^\delta)^\beta + CK_t^\eta (s - t_s^\delta)^r (1 + |X_{t_s^\delta}^{\delta,N}|) + |X_s^{\delta,N} - X_s^{\mu,N}| \right). \end{aligned}$$

Define

$$\|f\|_{R,t}^2 = \int_0^t \left(|f(s)| + \int_0^s |f(s) - f(z)| (s-z)^{-1-\alpha} dz \right)^2 g(t,s) \mathbb{I}_s ds$$

and $\Delta_t = \|X^{\delta,N} - X^{\mu,N}\|_{R,t}^2$.

Write

$$\|X^{\delta,N} - X^{\mu,N}\|_{R,t}^2 \leq 3(\|\mathcal{I}_a\|_{R,t}^2 + \|\mathcal{I}_b\|_{R,t}^2 + \|\mathcal{I}_c\|_{R,t}^2) \leq 6(I'_a + I''_a + I'_b + I''_b + I'_c + I''_c),$$

where $I'_d = \int_0^t |\mathcal{I}_d(s)|^2 g(t, s) \mathbb{I}_s ds$, $I''_d = \int_0^t \left(\int_0^s |\mathcal{I}_d(s) - \mathcal{I}_d(u)| h(s, u) du \right)^2 g(t, s) \mathbb{I}_s ds$, $d \in \{a, b, c\}$. We estimate this terms one by one. Note that some first estimates may be very rough. The reason is that we do not need them to be finer than the (apparently worse) estimates that follow.

We will need the following trivial formula, checked directly for $t > \tau_N$ and $t \leq \tau_N$:

$$(14) \quad \int_0^t \int_{s(N)}^{t(N)} f(t, s, v) dv ds = \int_0^{t(N)} \int_s^{t(N)} f(t, s, v) dv ds = \int_0^{t(N)} \int_0^v f(t, s, v) ds dv.$$

By (13) and the Cauchy–Schwartz inequality, we can write

$$\begin{aligned} I'_a &\leq C \int_0^t \int_0^{s(N)} (\delta^\beta + RN\delta^r + |X_u^{\delta, N} - X_u^{\mu, N}|)^2 du g(t, s) \mathbb{I}_s ds \\ &\leq C \int_0^t \int_0^s (\delta^{2\beta} + RN\delta^{2r} + |X_u^{\delta, N} - X_u^{\mu, N}|^2 \mathbb{I}_u) du g(t, s) ds \leq C_{R, N} \left(\delta^{2(\kappa-\eta)} + \int_0^t \Delta_s g(t, s) ds \right). \end{aligned}$$

Similarly,

$$\begin{aligned} I''_a &\leq C_{R, N} \int_0^t \left(\int_0^s \int_{u(N)}^{s(N)} (\delta^{\kappa-\eta} + |X_v^{\delta, N} - X_v^{\mu, N}|) \mathbb{I}_v dv h(s, u) du \right)^2 g(t, s) ds \\ &\leq C_{R, N} \int_0^t \left(\delta^{\kappa-\eta} + \int_0^s |X_v^{\delta, N} - X_v^{\mu, N}| (s-v)^{-\alpha} \mathbb{I}_v dv \right)^2 g(t, s) ds \\ &\leq C_{R, N} \left(\delta^{2(\kappa-\eta)} + \int_0^t \int_0^s |X_v^{\delta, N} - X_v^{\mu, N}|^2 (s-v)^{-\alpha} \mathbb{I}_v dv g(t, s) ds \right) \\ &\leq C_{R, N} \left(\delta^{2(\kappa-\eta)} + \int_0^t \Delta_s g(t, s) ds \right). \end{aligned}$$

Further, by (7)

$$\begin{aligned} I'_c &\leq CN \int_0^t \left[\int_0^{s(N)} \left(|c_\Delta(u)| u^{-\alpha} du + \int_0^u |c_\Delta(u) - c_\Delta(z)| h(u, z) dz \right) du \right]^2 \mathbb{I}_s g(t, s) ds \\ &\leq CN \int_0^t \left[\left(\int_0^s |c_\Delta(u)| u^{-\alpha} \mathbb{I}_u du \right)^2 + \left(\int_0^s \int_0^u |c_\Delta(u) - c_\Delta(z)| h(u, z) dz \mathbb{I}_u du \right)^2 \right] g(t, s) ds \\ &=: CN(J'_c + J''_c). \end{aligned}$$

Analogously to I'_a ,

$$J'_c \leq C_{R, N} \left(\delta^{2(\kappa-\eta)} + \int_0^t \Delta_s g(t, s) ds \right).$$

By [13, Lemma 7.1], the hypotheses (A)–(D) imply that for any $t_1, t_2, x_1, \dots, x_4$

$$(15) \quad \begin{aligned} |c(t_1, x_1) - c(t_2, x_2) - c(t_1, x_3) + c(t_2, x_4)| &\leq K |x_1 - x_2 - x_3 + x_4| \\ &+ K |x_1 - x_3| |t_2 - t_1|^\beta + K |x_1 - x_3| (|x_1 - x_2| + |x_3 - x_4|). \end{aligned}$$

Therefore, taking into account Hölder continuity of c in the first variable and Lipschitz continuity in the second, we get

$$\begin{aligned}
(16) \quad & |c_\Delta(u) - c_\Delta(z)| = |c(t_u^\delta, X_{t_u^\delta}^{\delta,N}) - c(t_z^\delta, X_{t_z^\delta}^{\delta,N}) - c(t_u^\mu, X_{t_u^\mu}^{\mu,N}) + c(t_z^\mu, X_{t_z^\mu}^{\mu,N})| \\
& \leq |c(u, X_u^{\delta,N}) - c(z, X_z^{\delta,N}) - c(u, X_u^{\mu,N}) + c(z, X_z^{\mu,N})| + C\left((u - t_u^\delta)^\beta + (z - t_z^\delta)^\beta\right. \\
& \quad \left. + (u - t_u^\mu)^\beta + (z - t_z^\mu)^\beta + |X_{u,t_u^\delta}^{\delta,N}| + |X_{z,t_z^\delta}^{\delta,N}| + |X_{u,t_u^\mu}^\mu| + |X_{z,t_z^\mu}^\mu|\right) \\
& \leq C_{R,N} \left(|X_u^{\delta,N} - X_z^{\delta,N} - X_u^{\mu,N} + X_z^{\mu,N}| + |X_u^{\delta,N} - X_u^{\mu,N}|(u - z)^\beta \right. \\
& \quad \left. + |X_u^{\delta,N} - X_u^{\mu,N}|(|X_{u,z}^{\delta,N}| + |X_{u,z}^{\mu,N}|) + (u - t_u^\delta)^{\kappa-\eta} + (z - t_z^\delta)^{\kappa-\eta} \right).
\end{aligned}$$

Note also that

$$|c_\Delta(u) - c_\Delta(z)| = |c(t_u^\mu, X_{t_u^\mu}^\mu) - c(t_z^\mu, X_{t_z^\mu}^\mu)| \leq C(|t_u^\mu - t_z^\mu|^\beta + |X_{t_u^\mu, t_z^\mu}^\mu|)$$

for $z \in [t_u^\delta, t_u^\mu]$ and $|c_\Delta(u) - c_\Delta(z)| = 0$ for $z \in [t_u^\mu, u]$, hence J_c'' admits an estimate

$$\begin{aligned}
J_c'' & \leq C_{R,N} \int_0^t \left[\int_0^s \int_0^{t_u^\delta} \left(|X_u^{\delta,N} - X_z^{\delta,N} - X_u^{\mu,N} + X_z^{\mu,N}| + |X_u^{\delta,N} - X_u^{\mu,N}|(u - z)^\beta \right. \right. \\
& \quad \left. \left. + |X_u^{\delta,N} - X_u^{\mu,N}|(|X_{u,z}^{\delta,N}| + |X_{u,z}^{\mu,N}|) \right. \right. \\
& \quad \left. \left. + (u - t_u^\delta)^{\kappa-\eta} + (z - t_z^\delta)^{\kappa-\eta} \right) h(u, z) dz du \right]^2 g(t, s) \mathbb{I}_s ds \\
& + \int_0^t \left(\int_0^s \int_{t_u^\delta}^{t_u^\mu} \left((t_u^\mu - t_z^\mu)^\beta + |X_{t_u^\mu, t_z^\mu}^{\mu,N}| \right) h(u, z) dz du \right)^2 g(t, s) \mathbb{I}_s ds \leq C_{R,N} (H_1 + H_2 + H_3 + H_4).
\end{aligned}$$

Here we split integrand simply by the rows to estimate them apart. Write

$$\begin{aligned}
H_1 & \leq C_{R,N} \int_0^t \left[\int_0^s \left(\int_0^u |X_u^{\delta,N} - X_z^{\delta,N} - X_u^{\mu,N} + X_z^{\mu,N}| h(u, z) dz \right)^2 \mathbb{I}_u du \right. \\
& \quad \left. + \left(\int_0^s |X_u^\delta - X_u^\mu| (u - t_u^\delta)^{\beta-\alpha} \mathbb{I}_u du \right)^2 \right] g(t, s) ds \leq C_{R,N} \int_0^t \Delta_s g(t, s) ds.
\end{aligned}$$

Further,

$$\begin{aligned}
H_2 & \leq C_{R,N} \int_0^t \left(\int_0^s |X_u^{\delta,N} - X_u^{\mu,N}| \int_0^u (|X_{u,z}^{\delta,N}| + |X_{u,z}^{\mu,N}|) h(u, z) dz \mathbb{I}_u du \right)^2 g(t, s) ds \\
& \leq C_{R,N} \int_0^t \left(\int_0^s |X_u^{\delta,N} - X_u^{\mu,N}| (\|X^{\delta,N}\|_{\infty, u} + \|X^{\mu,N}\|_{\infty, u}) \mathbb{I}_u du \right)^2 g(t, s) ds \\
& \leq C_{R,N} \int_0^t \int_0^s |X_u^{\delta,N} - X_u^{\mu,N}|^2 \mathbb{I}_u du g(t, s) ds \leq C_{R,N} \int_0^t \Delta_s g(t, s) ds.
\end{aligned}$$

Analogously to (10),

$$\begin{aligned}
H_3 &\leq C_{R,N} \int_0^t \left(\int_0^s \int_0^{t_u^\delta} ((u - t_u^\delta)^{\kappa-\eta} + (z - t_z^\delta)^{\kappa-\eta}) h(s, z) dz du \right)^2 g(t, s) \mathbb{I}_s ds \\
&\leq C_{R,N} \int_0^t \left[\left(\int_0^s (u - t_u^\delta)^{\kappa-\alpha-\eta} du \right)^2 + \int_0^t \left(\int_0^{t_s^\delta} (z - t_z^\delta)^{\kappa-\eta} (t_z^\delta + \delta - z)^{-\alpha} dz \right)^2 \right] g(t, s) ds \\
&\leq C_{R,N} \int_0^t \left[\delta^{2(\kappa-\alpha-\eta)} + \left(\sum_{k=1}^{[s/\delta]} \int_{\nu_{k-1}}^{\nu_k} (z - \nu_{k-1})^{\kappa-\eta} (\nu_k - z)^{-\alpha} dz \right)^2 \right] g(t, s) ds \\
&\leq C_{R,N} \left(\delta^{2(\kappa-\alpha-\eta)} + \int_0^t \left(\sum_{k=1}^{[s/\delta]} \delta \cdot \delta^{\kappa-\alpha-\eta} \right)^2 g(t, s) ds \right) \leq C_{R,N} \delta^{2(\kappa-\alpha-\eta)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
H_4 &\leq C \int_0^t \left(\int_0^s \int_{t_u^\delta}^{t_u^\mu} ((t_u^\mu - t_z^\mu)^\beta + RN(t_u^\mu - t_z^\mu)^r) h(u, z) dz du \right)^2 g(t, s) \mathbb{I}_s ds \\
&\leq C_{R,N} \int_0^t \left(\int_0^s \int_{t_u^\delta}^{t_u^\mu} ((u - z)^{\kappa-\eta} + (z - t_z)^{\kappa-\eta}) h(u, z) dz du \right)^2 g(t, s) ds \leq C_{R,N} \mu^{2(\kappa-\alpha-\eta)}.
\end{aligned}$$

Now turn to I_c'' . By (7) and (14), we can write

$$\begin{aligned}
I_c'' &\leq N \int_0^t \left(\int_0^{s(N)} \int_u^{s(N)} \left(|c_\Delta(v)| (v - u)^{-\alpha} \right. \right. \\
&\quad \left. \left. + \int_u^v |c_\Delta(v) - c_\Delta(z)| h(v, z) dz \right) dv h(s, u) du \right)^2 g(t, s) \mathbb{I}_s ds \\
&\leq C_N \int_0^t \left(\int_0^s \left(|c_\Delta(v)| (s - v)^{-2\alpha} + \int_0^v |c_\Delta(v) - c_\Delta(z)| h(v, z) (s - z)^{-\alpha} dz \right) \mathbb{I}_v dv \right)^2 g(t, s) ds \\
&\leq C_N (L'_c + L''_c),
\end{aligned}$$

where we have used that $\int_0^v (v - u)^{-\alpha} h(s, u) du \leq C(s - v)^{-2\alpha}$. The term L'_c is estimated similarly to I_a'' :

$$\begin{aligned}
L'_c &\leq C_{R,N} \int_0^t \left(\int_0^s \left(\delta^{\kappa-\eta} + |X_v^{\delta,N} - X_v^{\mu,N}| \right) (s - v)^{-2\alpha} \mathbb{I}_v dv \right)^2 g(t, s) ds \\
&\leq C_{R,N} \left(\delta^{2(\kappa-\eta)} + \int_0^t \int_0^s |X_v^{\delta,N} - X_v^{\mu,N}|^2 (s - v)^{-2\alpha} \mathbb{I}_v dv g(t, s) ds \right) \\
&\leq C_{R,N} \left(\delta^{2(\kappa-\eta)} + \int_0^t \Delta_s g(t, s) ds \right).
\end{aligned}$$

To handle L_c'' , we use the estimate (16) and proceed exactly as with J_c'' :

$$\begin{aligned} L_c'' &\leq C_{R,N} \int_0^t \left(\int_0^s \int_0^{t_v^\delta} \left(|X_v^{\delta,N} - X_z^{\delta,N} - X_v^{\mu,N} + X_z^{\mu,N}| + |X_v^{\delta,N} - X_v^{\mu,N}|(v-z)^\beta \right. \right. \\ &\quad \left. \left. + |X_v^{\delta,N} - X_v^{\mu,N}|(|X_{v,z}^{\delta,N}| + |X_{v,z}^{\mu,N}|) \right) \right. \\ &\quad \left. + (v-t_v^\delta)^{\kappa-\eta} + (z-t_z^\delta)^{\kappa-\eta} \right) h(v,z)(s-z)^{-\alpha} dz dv \Big)^2 \mathbb{I}_s g(t,s) ds \\ &+ \int_0^t \left(\int_0^s \int_{t_v^\delta}^{t_v^\mu} \left((t_v^\mu - t_z^\mu)^\beta + |X_{t_v^\mu, t_z^\mu}^{\mu,N}| \right) h(v,z)(s-z)^{-\alpha} dz dv \right)^2 \mathbb{I}_s g(t,s) ds \leq C_{R,N} \sum_{k=1}^4 G_k. \end{aligned}$$

Now

$$\begin{aligned} G_1 &\leq \int_0^t \left(\int_0^s \int_0^v |X_v^{\delta,N} - X_z^{\delta,N} - X_v^{\mu,N} + X_z^{\mu,N}| h(v,z) dz (s-v)^{-\alpha} \mathbb{I}_v dv \right)^2 g(t,s) ds \\ &\quad + \int_0^t \left(\int_0^s |X_v^{\delta,N} - X_v^{\mu,N}| \int_0^v (v-z)^{\beta-\alpha-1} (s-z)^{-\alpha} dz dv \mathbb{I}_v \right)^2 g(t,s) ds \\ &\leq \int_0^t \int_0^s \left(\int_0^v |X_v^{\delta,N} - X_z^{\delta,N} - X_v^{\mu,N} + X_z^{\mu,N}| h(v,z) dz \right)^2 (s-v)^{-\alpha} \mathbb{I}_v dv g(t,s) ds \\ &\quad + \int_0^t \left(\int_0^s |X_v^{\delta,N} - X_v^{\mu,N}| (s-v)^{\beta-2\alpha} \mathbb{I}_v dv \right)^2 g(t,s) ds \leq C \int_0^t \Delta_s g(t,s) ds. \end{aligned}$$

Further,

$$\begin{aligned} G_2 &\leq C \int_0^t \left(\int_0^s |X_v^{\delta,N} - X_v^{\mu,N}| (\|X^{\delta,N}\|_{\infty,v} + \|X^{\mu,N}\|_{\infty,v}) (s-v)^{-\alpha} \mathbb{I}_v dv \right)^2 g(t,s) ds \\ &\leq C_R \int_0^t \int_0^s |X_v^{\delta,N} - X_v^{\mu,N}|^2 (s-v)^{-\alpha} \mathbb{I}_v dv g(t,s) ds \leq C_R \int_0^t \Delta_s g(t,s) ds, \\ G_3 &\leq C \int_0^t \left(\int_0^s (v-t_v^\delta)^{\kappa-\alpha-\eta} (s-v)^{-\alpha} dv \right. \\ &\quad \left. + \int_0^{t_s^\delta} (z-t_z^\delta)^{\kappa-\eta} (s-z)^{-\alpha} (t_z^\delta + \delta - z)^{-\alpha} dz \right)^2 g(t,s) \mathbb{I}_s ds \leq C \delta^{2(\kappa-\eta-\alpha)}. \end{aligned}$$

Here the last integral is estimated analogously to the term H_3' (equation (10)) in the proof of Lemma 2.1:

$$\begin{aligned} &\int_0^{t_s^\delta} (z-t_z^\delta)^{\kappa-\eta} (s-z)^{-\alpha} (t_z^\delta + \delta - z)^{-\alpha} dz \\ &\leq \sum_{k=1}^{[s/\delta]-1} (s-\nu_k)^{-\alpha} \int_{\nu_{k-1}}^{\nu_k} (z-\nu_{k-1})^{\kappa-\eta} (\nu_k-z)^{-\alpha} dz + \delta^{\kappa-\eta} \int_{t_s^\delta-\delta}^{t_s^\delta} (s-z)^{-\alpha} (t_s^\delta-z)^{-\alpha} dz \\ &\leq C \left(\sum_{k=1}^{[s/\delta]-1} (s-\nu_k)^{-\alpha} \delta \cdot \delta^{\kappa-\alpha-\eta} + \delta^{\kappa-\alpha} \delta^{1-2\alpha} \right) \leq C \delta^{k-\alpha-\eta} \left(\int_0^s (s-z)^{-\alpha} dz + 1 \right) \leq C \delta^{k-\alpha-\eta}. \end{aligned}$$

Similarly,

$$\begin{aligned}
G_4 &\leq C \int_0^t \left(\int_0^s \int_0^{t_v^\mu} \left((t_v^\mu - t_z^\mu)^\beta + RN(t_v^\mu - t_z^\mu)^r \right) h(v, z) (s - z)^{-\alpha} dz dv \right)^2 g(t, s) ds \\
&\leq C_{R,N} \int_0^t \left(\int_0^s \int_0^{t_v^\mu} \left((v - z)^{\kappa-\eta} + (z - t_z^\mu)^{\kappa-\eta} \right) h(v, z) dz (s - z)^{-\alpha} dz dv \right) g(t, s) ds \\
&\leq C_{R,N} \mu^{2(\kappa-\alpha-\eta)}.
\end{aligned}$$

Summing up, we have an estimate

$$\|\mathcal{I}_a\|_{R,t}^2 + \|\mathcal{I}_c\|_{R,t}^2 \leq C_{R,N} \left(\delta^{2(\kappa-\alpha-\eta)} + \int_0^t \Delta_s g(t, s) ds \right).$$

Hence,

$$(17) \quad E \left[\left(\|\mathcal{I}_a\|_{R,t} + \|\mathcal{I}_c\|_{R,t} \right)^2 \mathbb{I}_{B_t^R} \right] \leq C_R \left(\delta^{2(\kappa-\alpha-2\eta)} + \int_0^t E[\Delta_s] g(t, s) ds \right).$$

Now turn to I'_b and I''_b . Using (13) and noting that $B_s^R \subset B_u^R \in \mathcal{F}_u$ for $u < s$, we have

$$\begin{aligned}
E[I'_b] &= \int_0^t E \left[\left(\int_0^{s(N)} b_\Delta(u) dW(u) \right)^2 \mathbb{I}_s \right] g(t, s) ds \\
&\leq \int_0^t \int_0^s E [b_\Delta(u)^2 \mathbb{I}_u] du g(t, s) ds \\
&\leq C \int_0^t \int_0^s E \left[(\delta^\beta + RN\delta^r + |X_u^{\delta,N} - X_u^{\mu,N}|)^2 \mathbb{I}_u \right] du g(t, s) ds \\
&\leq C_{R,N} \left(\delta^{2(\kappa-\eta)} + \int_0^t E[\Delta_s] g(t, s) ds \right).
\end{aligned}$$

Further,

$$\begin{aligned}
E[I''_b] &= \int_0^t E \left[\left(\int_0^{s(N)} \left| \int_u^{s(N)} b_\Delta(v) dW_v \right| (s - u)^{-1-\alpha} ds \right)^2 \mathbb{I}_s \right] g(t, s) ds \\
&\leq \int_0^t \int_0^s E \left[\left(\int_u^{s(N) \vee u} b_\Delta(v) dW_v \right)^2 \mathbb{I}_{B_s^R} \right] (s - u)^{-3/2-\alpha} du \int_0^s (s - u)^{-1/2-\alpha} du g(t, s) ds \\
&\leq C \int_0^t \int_0^s \int_u^s E \left[(\delta^\beta + RN\delta^r + |X_v^{\delta,N} - X_v^{\mu,N}|)^2 \mathbb{I}_v \right] dv (s - u)^{-3/2-\alpha} du g(t, s) ds \\
&\leq C_{R,N} \left(\delta^{2(\kappa-\eta)} + \int_0^t \int_0^s E[|X_v^{\delta,N} - X_v^{\mu,N}|^2 \mathbb{I}_v] (s - v)^{-1/2-\alpha} dv g(t, s) ds \right) \\
&\leq C_{R,N} \left(\delta^{2(\kappa-\eta)} + \int_0^t E[\Delta_s] g(t, s) ds \right).
\end{aligned}$$

Combining this with (17), we get

$$\begin{aligned}
E[\Delta_t] &\leq C_{R,N} \left(\delta^{2(\kappa-\alpha-\eta)} + \int_0^t E[\Delta_s] g(t, s) ds \right) \\
&\leq C_{R,N} \left(\delta^{2(\kappa-\alpha-\eta)} + t^{1/2+\alpha} \int_0^t E[\Delta_s] (t - s)^{-1/2-\alpha} s^{-1/2-\alpha} ds \right),
\end{aligned}$$

whence by the generalized Gronwall lemma

$$E[\Delta_t] \leq C_{R,N} \delta^{2(\kappa-\alpha-\eta)}.$$

Obviously, $\|f\|_{2,T} \mathbb{1}_T \leq \|f\|_{R,T}$, so the first statement of the theorem is proved.

To prove the second, for we derive for $\omega \in B_s^R$ absolutely similarly to the previous estimates (it is easy to check that the terms δ^a with $a > 0$ enter with bounded coefficients *before* the integration with respect to s) that

$$\mathcal{I}_a(s)^2 + \mathcal{I}_c(s)^2 \leq C_{R,N} (\delta^{2(\kappa-\alpha-\eta)} + \Delta_s).$$

The remaining term $\mathcal{I}_b(s)$ estimated similarly with the help of Burkholder inequality:

$$E \left[\sup_{u \leq s} \mathcal{I}_b(u)^2 \mathbb{1}_s \right] \leq C_{R,N} (\delta^{2(\kappa-\eta)} + E[\Delta_s]),$$

whence

$$E \left[\sup_{s \leq T} (\mathcal{I}_a(s)^2 + \mathcal{I}_b(s)^2 + \mathcal{I}_c(s)^2) \mathbb{1}_s \right] \leq C_{R,N} (\delta^{2(\kappa-\alpha-\eta)} + E[\Delta_T]) \leq C_{R,N} (\delta^{2(\kappa-\alpha-\eta)}),$$

but $(X_s^{\delta,N} - X_s^{\mu,N})^2 \leq 3(\mathcal{I}_a(s)^2 + \mathcal{I}_b(s)^2 + \mathcal{I}_c(s)^2)$, so the second statement follows. \square

4. EXISTENCE AND UNIQUENESS

Theorem 4.1. *Equation (1) has a solution such that for any $\alpha \in (1 - H, \kappa)$*

$$(18) \quad \|X\|_{\infty, \alpha, [0, T]}^2 < \infty \quad a.s.$$

This solution is unique in the class of processes satisfying (18) for some $\alpha > 1 - H$.

Remark 4.1. It is possible to generalize this result almost verbatim to a multidimensional case. Moreover, instead of fractional Brownian motion one can take any process, which is almost surely Hölder continuous with Hölder exponent $\gamma > 1/2$.

Proof. As in the previous proof, we denote $Z_{t,s} = Z_t - Z_s$, $r = 1/2 - \eta$, $h(t, s) = (t - s)^{-1-\alpha}$, $g(t, s) = s^{-\alpha} + (t - s)^{-1/2-\alpha}$. C is a generic constant, which may depend on fixed parameters, but is independent of variables.

Existence. Let $\delta_k = T/2^k$. To keep notations simple, denote $X^k = X^{\delta_k}$, $X^{k,N} = X^{\delta_k, N}$, $t_s^k = t_s^{\delta_k}$, $\|\cdot\| = \|\cdot\|_{2,T}$, $\|\cdot\|_\infty = \|\cdot\|_\infty$.

It is easy to see that for each $N \geq 1$ the sequence $\{X_t^{k,N}, k \geq 1\}$ is fundamental in the norm $(E[\|\cdot\|^2])^{1/2}$. Indeed, define $A^{R,N,k,l} = \{\|X^{k,N}\| + \|X^{l,N}\| \leq R\}$, take $p > 1$, denote $q = p/(p-1)$ and write

$$\begin{aligned} & E \left[\|X^{k,N} - X^{l,N}\|^2 \right] \\ & \leq E \left[\|X^{k,N} - X^{l,N}\|^2 \mathbb{1}_{A^{R,N,k,l}} \right] + E \left[(\|X^{k,N}\|_\infty + \|X^{l,N}\|_\infty)^2 \mathbb{1}_{\Omega \setminus A^{R,N,k,l}} \right] \\ & \leq E \left[\|X^{k,N} - X^{l,N}\|^2 \mathbb{1}_{A^{R,N,k,l}} \right] + \left(E \left[(\|X^{k,N}\|_\infty + \|X^{l,N}\|_\infty)^{2p} \right] \right)^{1/p} P(\Omega \setminus A^{R,N,k,l})^{1/q}. \end{aligned}$$

By Theorem 3.1, the first term vanishes as $k, l \rightarrow \infty$, so we can write

$$\overline{\lim}_{k,l \rightarrow \infty} E \left[\|X^{k,N} - X^{l,N}\|^2 \right] \leq \sup_{k,l} \left(E \left[(\|X^{k,N}\|_\infty + \|X^{l,N}\|_\infty)^{2p} \right] \right)^{1/p} P(\Omega \setminus A^{R,N,k,l})^{1/q}.$$

Lemma 2.1 implies that $\sup_{k,l} E \left[(\|X^{k,N}\|_\infty + \|X^{l,N}\|_\infty)^{2p} \right] < \infty$ and also, through Markov inequality, that $\sup_{k,l} P(\Omega \setminus A^{R,N,k,l}) \rightarrow 0, R \rightarrow \infty$. Therefore, letting $R \rightarrow \infty$, we get

$$\overline{\lim}_{k,l \rightarrow \infty} E \left[\|X^{k,N} - X^{l,N}\|^2 \right] = 0,$$

as claimed. Similarly, from the second statement of Theorem 3.1 we obtain

$$\overline{\lim}_{k,l \rightarrow \infty} E \left[\sup_{t \in [0,T]} |X^{k,N} - X^{l,N}|^2 \right] = 0.$$

Thus, for each $N \geq 1$ there exists a process X^N such that $E \left[\|X^{k,N} - X^N\|^2 \right] \rightarrow 0$ and $E \left[\sup_{t \in [0,T]} |X^{k,N} - X^N|^2 \right] \rightarrow 0, k \rightarrow \infty$; the limits agree because each of this two facts implies the convergence in $L^2([0,T] \times \Omega)$. Using a usual argument we can show that there exists a subsequence $\{k_j, j \geq 1\}$ such that for any $N \geq 1$ $\|X^{k_j,N} - X^N\| + \sup_{t \in [0,T]} |X^{k_j,N} - X^N| \rightarrow 0$ a.s., $j \rightarrow \infty$. Without loss of generality we can assume that the sequence itself converges a.s. to zero:

$$(19) \quad \|X^{k,N} - X^N\| + \sup_{t \in [0,T]} |X^{k,N} - X^N| \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty.$$

Note that for each $N \geq 1, p > 0$ $E[\|X^N\|_\infty^p] < \infty$. Indeed, we already have a uniform convergence, so it is enough to show boundedness of the integral in the definition of $\|\cdot\|_\infty$. By the Fatou lemma,

$$\int_0^t |X_{t,z}^N| (t-z)^{-1-\alpha} dz \leq \underline{\lim}_{k \rightarrow \infty} \int_0^t |X_{t,z}^{k,N}| (t-z)^{-1-\alpha} dz \leq \underline{\lim}_{k \rightarrow \infty} \|X^{k,N}\|_\infty,$$

therefore

$$\sup_{s \in [0,T]} \int_0^t |X_{t,z}^N| (t-z)^{-1-\alpha} dz \leq \underline{\lim}_{k \rightarrow \infty} \|X^{k,N}\|_\infty,$$

and applying the Fatou lemma again, we get

$$E \left[\left(\sup_{s \in [0,T]} \int_0^t |X_{t,z}^N| (t-z)^{-1-\alpha} dz \right)^p \right] \leq \underline{\lim}_{k \rightarrow \infty} E[\|X^{k,N}\|_\infty^p] < \infty.$$

Further, for $N' \leq N''$ and $t \leq \tau_{N'}$ $X_t^{N'} = X_t^{N''}$ a.s. Indeed, for any t $X_t^{k,N'} \rightarrow X_t^{N''}$ and $X_t^{k,N''} \rightarrow X_t^{N''}$ a.s. as $k \rightarrow \infty$; but for $t \leq \tau_{N'}$ $X_t^{k_j,N'} = X_t^{k_j,N''}$, so $X_t^{N'} = X_t^{N''}$ a.s.

Hence there exists a process X such that for any N and any t $X_t^N = X_{t \wedge \tau_N}$. We are going to prove that X solves (1). This will be done by showing that each of the integrals in (8) converges to a corresponding integral in (1). To this end, consider the differences between these integrals:

$$\mathcal{I}_a^N(t) = \int_0^{t(N)} a_\Delta(s) ds, \quad \mathcal{I}_b^N(t) = \int_0^{t(N)} b_\Delta(s) dW_s, \quad \mathcal{I}_c^N(t) = \int_0^{t(N)} c_\Delta(s) dB_s^H,$$

here $d_\Delta(s) = d(s, X_s) - d(t_s^k, X_{t_s^k}^k), d \in \{a, b, c\}$. Similarly to (13),

$$(20) \quad |d_\Delta| \leq C \left((s - t_s^k)^\beta + CK_t^\eta (s - t_k^\delta)^r (1 + |X_{t_s^k}^{k,N}|) + |X_s^{k,N} - X_s| \right)$$

Estimate

$$|\mathcal{I}_a^N(t)| \leq \int_0^t |a_\Delta(s(N))| ds \leq C \left(\delta_k^\beta + N \delta_k^r (1 + \|X^{k,N}\|) + \|X^{k,N} - X^N\| \right).$$

Thanks to (19), the norm $\|X^{k,N}\|$ is a.s. bounded in k . Consequently, $\mathcal{I}_a^N(t) \rightarrow 0$ a.s., $k \rightarrow \infty$.

Further,

$$\begin{aligned} E \left[(\mathcal{I}_b^N(t))^2 \right] &\leq \int_0^t E [b_\Delta(s)^2] ds \\ &\leq C \left(\delta_k^\beta + \delta_k^r E \left[(1 + \|X^{k,N}\|)^2 \right] + E \left[\|X^{k,N} - X^N\|^2 \right] \right) \rightarrow 0, k \rightarrow \infty. \end{aligned}$$

We can extract a subsequence $\{k_j, j \geq 1\}$ such that for each $N \geq 1$ $\mathcal{I}_a^N(t) \rightarrow 0$ a.s., $j \rightarrow \infty$. Again, we will assume without loss of generality that sequence itself vanishes.

Finally,

$$|\mathcal{I}_c^N(t)| \leq CN \int_0^t \left(|c_\Delta(s)| s^{-\alpha} ds + \int_0^s |c_\Delta(s) - c_\Delta(z)| h(s, z) dz \right) ds =: CN(I'_c + I''_c).$$

Similarly to $\mathcal{I}_a^N(t)$, $I'_c \rightarrow 0$, $k \rightarrow \infty$. Using an estimate similar to (16), write

$$\begin{aligned} &\int_0^t \int_0^{t_s^\delta} \left(|X_s^N - X_z^N - X_s^{k,N} + X_z^{k,N}| + |X_s^N - X_s^{k,N}| (s-z)^\beta \right. \\ &+ |X_s^N - X_s^{k,N}| \left(|X_{s,z}^N| + |X_{s,z}^{k,N}| \right) + (s-t_s^k)^\beta + (z-t_k^\delta)^\beta + |X_{s,t_s^k}^{k,N}| + |X_{z,t_z^k}^{k,N}| \Big) h(s, z) dz ds \\ &\quad + \int_0^t \int_{t_s^\delta}^s \left((s-z)^\beta + |X_{s,z}^k| \right) h(s, z) dz ds \\ &\leq CN \left(\|X^{k,N} - X^N\| (1 + \|X^N\|_\infty + \|X^{k,N}\|_\infty) + \delta_k^\beta \right. \\ &\quad \left. + (1 + \|X^{k,N}\|) \int_0^t \int_0^{t_s^\delta} ((s-t_s^\delta)^r + (z-t_z^\delta)^r) h(s, z) dz ds + \delta_k^{\beta-\alpha} + \delta_k^{r-\alpha} (1 + \|X^{k,N}\|) \right) \\ &\leq CN \left(\|X^{k,N} - X^N\| (1 + \|X^N\|_\infty + \|X^{k,N}\|_\infty) + \delta_k^{\beta-\alpha} + \delta_k^{r-\alpha} (1 + \|X^{k,N}\|_\infty) \right) \rightarrow 0, k \rightarrow \infty. \end{aligned}$$

The last holds since $\|X^N\|_\infty$ is almost surely finite and $\|X^{k,N}\|_\infty$ is bounded in probability uniformly in k .

As a result, $|\mathcal{I}_a^N(s)| + |\mathcal{I}_b^N(s)| + |\mathcal{I}_c^N(s)| \rightarrow 0$, $k \rightarrow \infty$. But since $X_t^N - X_t^{k,N} = \mathcal{I}_a^N(t) + \mathcal{I}_b^N(t) + \mathcal{I}_c^N(t)$ and X^k solves (8) with $\delta = \delta_k$, we get that X solves (1) up to each of the moments τ_N . But (since K_T^η is finite a.s.) $\tau_N \rightarrow T$ a.s., $N \rightarrow \infty$, which implies that X is a solution to (1).

Uniqueness. Let X^1, X^2 be two solutions of (1). Define a stopping time

$$\sigma_N = \inf \left\{ t : K_t^\eta + \|X^1\|_{\alpha,t} + \|X^2\|_{\alpha,t} \geq N \right\}$$

and denote $X_t^{1,N} = X_{t \wedge \sigma_N}^1$, $X_t^{2,N} = X_{t \wedge \sigma_N}^2$, $\Delta_s = \|X^{1,N} - X^{2,N}\|_{2,s}^2$.

We have

$$\begin{aligned} X_u^{1,N} - X_u^{2,N} &= \int_0^{u(N)} a_\Delta(s) ds + \int_0^{u(N)} b_\Delta(s) dW_s + \int_0^{u(N)} c_\Delta(s) dB_s^H \\ &=: \mathcal{I}_a(u) + \mathcal{I}_b(u) + \mathcal{I}_c(u), \end{aligned}$$

where $d_\Delta(s) = d(s, X_s^{1,N}) - d(s, X_s^{2,N})$, $d \in \{a, b, c\}$, we have $|d_\Delta(s)| \leq C |X_s^{1,N} - X_s^{2,N}|$. Thus,

$$\Delta_t^2 \leq 3(\|\mathcal{I}_a\|_{2,t}^2 + \|\mathcal{I}_b\|_{2,t}^2 + \|\mathcal{I}_c\|_{2,t}^2).$$

The estimates will be similar to those of Theorem 3.1, so we skip some minor details.

Estimate

$$\begin{aligned} \|\mathcal{I}_a\|_{2,t}^2 &\leq \int_0^t \left(\int_0^s |a_\Delta(u)| du ds + \int_0^s \int_u^s |a_\Delta(v)| dv h(s, u) du \right)^2 g(t, s) ds \\ &\leq C \int_0^t \left[\Delta_s + \left(\int_0^s |X_v^{1,N} - X_v^{2,N}| (s-v)^{-\alpha} dv \right)^2 \right] g(t, s) ds \leq C \int_0^t \Delta_s g(t, s) ds. \end{aligned}$$

Further, $\|\mathcal{I}_c(t)\|_{2,t}^2 \leq \int_0^t [\mathcal{I}_c(s)^2 + (\int_0^s |\mathcal{I}_c(s) - I_c(u)| h(s, u) du)^2] g(t, s) ds =: I'_c + I''_c$. Using (7) and (15), write

$$\begin{aligned} I'_c &\leq CN \int_0^t \left[\int_0^s \left(|c_\Delta(u)| u^{-\alpha} + \int_0^u |c_\Delta(u) - c_\Delta(z)| h(u, z) dz \right) du \right]^2 g(t, s) ds \\ &\leq CN \int_0^t \Delta_s g(t, s) ds + CN \int_0^t \int_0^s \left(\int_0^u (|X_u^{1,N} - X_z^{1,N} - X_u^{2,N} + X_z^{2,N}| \right. \\ &\quad \left. + (u-z)^\beta |X_u^{1,N} - X_u^{2,N}| + |X_u^{1,N} - X_u^{2,N}| (|X_{u,z}^{1,N}| + |X_{u,z}^{2,N}|) \right) h(s, z) dz \Big)^2 du g(t, s) ds \\ &\leq CN \int_0^t \Delta_s g(t, s) ds. \end{aligned}$$

Similarly, by (7) and (15)

$$\begin{aligned} I''_c &\leq CN \int_0^t \left[\int_0^s \int_0^s \left(|c_\Delta(v)| (v-u)^{-\alpha} + \int_u^v |c_\Delta(v) - c_\Delta(z)| h(v, z) dz \right) dv h(s, u) du \right]^2 g(t, s) ds \\ &\leq CN \int_0^t \left[\int_0^s \left(|X_v^{1,N} - X_v^{2,N}| (s-v)^{-2\alpha} + \int_0^v |c_\Delta(v) - c_\Delta(z)| h(v, z) dz \right) (s-v)^{-\alpha} dv \right]^2 g(t, s) ds \\ &\leq CN \int_0^t \Delta_s g(t, s) ds + CN \int_0^t \int_0^s \left(\int_0^v (|X_{v,z}^{1,N} - X_{v,z}^{2,N}| + |X_v^{1,N} - X_v^{2,N}| (v-z)^\beta \right. \\ &\quad \left. + |X_v^{1,N} - X_v^{2,N}| (|X_{v,z}^{1,N}| + |X_{v,z}^{2,N}|) \right) h(v, z) dz \Big)^2 (s-v)^{-\alpha} dv g(t, s) ds \leq CN \int_0^t \Delta_s g(t, s) ds. \end{aligned}$$

Summing the estimates for $\mathcal{I}_a(t)$ and $\mathcal{I}_c(t)$ and using the Cauchy–Schwartz inequality, we get

$$E \left[(\|\mathcal{I}_a(t)\|_{2,t} + \|\mathcal{I}_c(t)\|_{2,t})^2 \right] \leq CN \int_0^t E[\Delta_s] g(t, s) ds.$$

Finally,

$$\begin{aligned}
E[\|\mathcal{I}_b(t)\|_{2,t}^2] &\leq 2 \int_0^t E[|\mathcal{I}_b(s)|^2] g(t,s) ds + 2 \int_0^t E\left[\left(\int_0^{s(N)} \left|\int_u^{s(N)} b_v dW_v\right| h(s,u) du\right)^2\right] g(t,s) ds \\
&\leq C \int_0^t \int_0^s E[b_\Delta(u)^2] du g(t,s) ds + C \int_0^t \int_0^s \int_u^s E[b_\Delta(v)^2] dv (s-u)^{-3/2-\alpha} du g(t,s) ds \\
&\leq C \int_0^t E[\Delta_s] g(t,s) ds.
\end{aligned}$$

As a result,

$$E[\Delta_t] \leq C_N \int_0^t E[\Delta_s] g(t,s) ds,$$

so by the generalized Gronwall lemma $E[\Delta_t] = 0$ for all t . This implies $X_t^1 = X_t^2$ a.s. for all $t < \tau_N$. By letting $N \rightarrow \infty$ we get $X_t^1 = X_t^2$ a.s. for all t . \square

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